TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 2, February 1999, Pages 581–594
S 0002-9947(99)01988-1

GOLUBEV SERIES FOR SOLUTIONS OF ELLIPTIC EQUATIONS

CH. DORSCHFELDT AND N. N. TARKHANOV

Abstract. Let P be an elliptic system with real analytic coefficients on an open set $X \subset \mathbb{R}^n$, and let Φ be a fundamental solution of P. Given a locally connected closed set $\sigma \subset X$, we fix some massive measure m on σ. Here, a non-negative measure m is called massive, if the conditions $s \subset \sigma$ and $m(s) = 0$ imply that $\overline{\sigma \setminus s} = \sigma$. We prove that, if f is a solution of the equation $Pf = 0$ in $X \setminus \sigma$, then for each relatively compact open subset U of X and every $1 < p < \infty$ there exist a solution f_e of the equation in U and a sequence f_α $(\alpha \in \mathbb{N}_0^n)$ in $L^p(\sigma \cap U, m)$ satisfying $\|\alpha! f_\alpha\|^{1/|\alpha|}_{L^p(\sigma\cap U,m)} \to 0$ such that $f(x) = f_e(x) + \sum_\alpha \int_{\sigma\cap U} D_y^\alpha \Phi(x,y) f_\alpha(y) dm(y)$ for $x \in U \setminus \sigma$. This complements an earlier result of the second author on representation of solutions outside a compact subset of X.

1. Introduction and statement of the main results

1.1. Let P be a $(k \times k)$-matrix of scalar partial differential operators with real analytic coefficients on an open set $X \subset \mathbb{R}^n$. Suppose further that P has a fundamental solution Φ which is real analytic outside the diagonal Δ of $X \times X$. By definition, $\Phi(x, y)$ is a $(k \times k)$-matrix of distributions on $X \times X$ satisfying

$$\begin{cases} P(x, D_x)\,\Phi(x, y) &= \delta(x - y)\, I_k, \\ P'(y, D_y)\,\Phi(x, y) &= \delta(x - y)\, I_k \end{cases}$$

where P' is the transposed operator to P, and I_k is the identity $(k \times k)$-matrix.

Recall that, according to a theorem of Malgrange, every elliptic differential operator with real analytic coefficients on X has a fundamental solution with the desired properties.

1.2. If U is an open subset of X, then denote by $S_P(U)$ the vector space of all weak solutions of the system $Pf = 0$ on U. Note that because of the analytic hypoellipticity of P, the solutions in $S_P(U)$ are actually real analytic functions in U. For a closed subset σ of X, solutions $f \in S_P(X \setminus \sigma)$ will be said to have singularities on σ.

In this article, we are interested in representations of solutions of the equation $Pf = 0$ in X having singularities on a closed subset σ of X. Before stating our principal result, we must first introduce one technical definition.

A (nonnegative) measure m on σ is said to be *massive*, if the two conditions $s \subset \sigma$ and $m(s) = 0$ imply that $\overline{\sigma \setminus s} = \sigma$. In other words, every subset of σ of

Received by the editors February 15, 1995 and, in revised form, November 20, 1996.
1991 *Mathematics Subject Classification.* Primary 35A20, 35C10.
Key words and phrases. Solutions with singularities, real analytic coefficients, elliptic systems, Golubev series.
This research was supported in part by the Alexander von Humboldt Foundation.

m-measure zero has empty interior. As the following example shows, a massive measure exists on every closed set σ.

Example 1.1. Let $\{y_j\}_{j\in\mathbb{N}}$ be a sequence of points of K, which is dense as a set in σ. Choose a sequence of positive numbers $\{\mu_j\}$ such that $\sum \mu_j < \infty$. For a set $s \subset \sigma$, we define $m(s) = \sum_{y_\nu \in s} \mu_\nu$. Then m is a massive measure on σ.

Let us fix some massive measure m on σ. Our main result is the following:

Theorem 1.1. *Assume that K is a locally connected compact subset of σ, and $1 < p < \infty$. Then for each solution $f \in S_P(X \setminus \sigma)$ there exist both a solution $f_e \in S_P((X \setminus \sigma) \cup \overset{\circ}{K})$ and a sequence $\{c_\alpha\}_{\alpha\in\mathbb{N}_0^n} \subseteq [L^p(K,m)]^k$ such that*

$$f(x) \;=\; f_e(x) + \sum_{\alpha\in\mathbb{N}_0^n} \int_K D_y^\alpha \Phi(x,y) c_\alpha(y) dm(y) \tag{1}$$

holds for all $x \in X \setminus \sigma$. Furthermore, $\|\alpha!\, c_\alpha\|_{L^p(K,m)}^{1/|\alpha|} \to 0$ when $|\alpha| \to \infty$.

We emphasize that $\overset{\circ}{K}$ is the interior of K on σ, i.e., in the induced topology of σ.

1.3. For holomorphic functions of one variable (i.e. for the operator $P = \partial/\partial\overline{z}$ in $\mathbb{C}^1$), for compact σ and $p = 2$, Theorem 1.1 is due to Havin [7]. Havin called the corresponding representation of the form (1) *Golubev-series*, since it was V.V. Golubev who posed the question whether such a formula held for every function analytic in $\hat{\mathbb{C}} \setminus K$ when K is a rectifiable simple arc and m the Lebesgue measure on K. For further details on the history of the problem cf. Havin [8]. More generally, we call representations of the form (1) *Golubev-series expansions* for solutions with singularities.

Baernstein [1] proved an analogous representation formula for functions holomorphic off the real axis. Using complex analysis and Hilbert space methods, the second author [15] showed Theorem 1.1 for the case of compact σ and $p = 2$ (see also [16]). Simonova [13] obtained an analogous representation theorem for functions harmonic off a hyperplane. Fischer and Tarkhanov [4] constructed a Golubev-series expansion for solutions of homogeneous elliptic systems with constant coefficients in $\mathbb{R}^n$, having singularities on a plane of a smaller dimension. They also derived Theorem 1.1 for the case of smooth σ and asked whether a result as formulated in Theorem 1.1 held for arbitrary locally connected sets σ.

The local connectedness of the compact set K we look at is a very delicate point in the literature. In fact it is related to the problem of extension of analytic functions on a neighborhood of K. (See Havin [8], Varfolomeev [17] and Rogers/Zame [12].)

In this paper, we prove the result by generalizing the ideas used in [15] in an appropriate way. Since the article [15] is in Russian and does not seem to be easily available, we have decided to present the paper in a self-contained way and do not use [15] as a reference.

1.4. The converse statement to Theorem 1.1 is quite easy to prove.

Lemma 1.1. *Let K be a relatively compact subset of σ, and $1 \le p < \infty$. For every sequence $\{c_\alpha\}_{\alpha\in\mathbb{N}_0^n} \subset [L^p(K,m)]^k$, satisfying $\|\alpha! c_\alpha\|_{L^p(K,m)}^{1/|\alpha|} \to 0$ when $|\alpha| \to \infty$, the series $\sum_\alpha \int_K D_y^\alpha \Phi(x,y) c_\alpha(y) dm(y)$ converges for $x \in X \setminus K$ and defines an element in $S_P(X \setminus K)$.*

Proof. First note that for $x \in X \setminus K$ we have

$$P(x,D)\int_K D_y^\alpha \Phi(x,y)c_\alpha(y)dm = \int_K D_y^\alpha\{P(x,D)\Phi(x,y)\}c_\alpha(y)dm = 0.$$

Thus the proof will be complete if we show that the series we look at converges uniformly on compact subsets of $X \setminus K$. It is well-known that a C^∞ function g on an open set $U \subset \mathbb{R}^n$ is real analytic if and only if for every compact set $K \subset U$ there are constants $a = a(g,K)$ and $c = c(g,K)$ such that

$$\sup_{y\in K} |D^\alpha g(y)| \le c\cdot a^{|\alpha|}|\alpha|! \text{ for all } \alpha \in \mathbb{N}_0^n.$$

Now fix a compact set $\tilde{K} \subset\subset X \setminus K$. Since the fundamental solution Φ is real analytic in a neighborhood of $\tilde{K}\times K$, there exist constants a and c, depending on Φ and $\tilde{K}$, such that

$$\sup_{(x,y)\in\tilde{K}\times K} \|D_y^\alpha\Phi(x,y)\| \le c\cdot a^{|\alpha|}|\alpha|! \text{ for all } \alpha\in\mathbb{N}_0^n. \tag{2}$$

Using (2), for $\alpha \in \mathbb{N}_0^n$ we get

$$\begin{aligned}\sup_{x\in\tilde{K}} \left|\int_K D_y^\alpha\Phi(x,y)c_\alpha(y)dm(y)\right| &\le c\cdot a^{|\alpha|}|\alpha|!\int_K |c_\alpha(y)|dm(y)\\ &\le c\cdot a^{|\alpha|}|\alpha|!\,\|c_\alpha\|_{L^p(K,m)}m(K)^{\frac{1}{q}},\end{aligned}$$

with $p^{-1}+q^{-1}=1$. Therefore

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}_0^n}\sup_{x\in\tilde{K}}\left|\int_K D_y^\alpha\Phi(x,y)c_\alpha(y)dm\right| &\le c\cdot m(K)^{1/q}\sum_{\alpha\in\mathbb{N}_0^n} a^{|\alpha|}|\alpha|!\,\|c_\alpha\|_{L^p(K,m)}\\ &= c\cdot m(K)^{1/q}\sum_{j=0}^\infty a^j\sup_{|\alpha|=j}\|\alpha! c_\alpha\|_{L^p(K,m)}\Big(\sum_{|\alpha|=j}\frac{|\alpha|!}{\alpha!}\Big)\\ &= c\cdot m(K)^{1/q}\sum_{j=0}^\infty\Big(a\cdot n\sup_{|\alpha|=j}\|\alpha!\,c_\alpha\|^{1/|\alpha|}_{L^p(K,m)}\Big)^j,\end{aligned}$$

where we used that $\sum_{|\alpha|=j}\frac{|\alpha|!}{\alpha!} = n^j,\ \ n=\dim\mathbb{R}^n$.

Now, since $\sup_{|\alpha|=j}\|\alpha!\,c_\alpha\|^{1/|\alpha|}_{L^p(K,m)} \to 0$ when $j\to\infty$, the last sum can be majorized by a geometric sum. Hence

$$\sum_{\alpha\in\mathbb{N}_0^n}\sup_{x\in\tilde{K}}\left|\int_K D_y^\alpha\Phi(x,y)c_\alpha(y)dm\right| \le c(K,\tilde{K}) < \infty.$$

□

1.5. Let us distinguish the principal difficulty in the proof of Theorem 1.1.

Lemma 1.2. *Let K be a locally connected compact subset of X, m be a massive measure on K and $1<p<\infty$. Then for every solution $f\in S_P(X\setminus K)$ there are a solution $f_e \in S_P(X)$ and a sequence $\{c_\alpha\}_{\alpha\in\mathbb{N}_0^n}\subset[L^p(K,m)]^k$ such that*

$$f(x) = f_e(x) + \sum_{\alpha\in\mathbb{N}_0^n}\int_K D_y^\alpha\Phi(x,y)c_\alpha(y)dm(y)$$

holds for all $x\in X\setminus K$. Furthermore, $\|\alpha! c_\alpha\|^{1/|\alpha|}_{L^p(K,m)} \to 0$ when $|\alpha|\to\infty$.

As Baernstein showed in [2], even for $P = \partial/\partial\overline{z}$ Lemma 1.2 is false for arbitrary compact K.

We now turn to the

Proof (of Theorem 1.1). Let $U \subset X$ be a relatively compact open set such that $U \cap \sigma = \overset{\circ}{K}$ and the set $K' = \partial U \cup \overset{\circ}{K}$ is locally connected. Fix some massive measure m' on K' whose restriction to K is m. The existence of such a measure follows from Example 1.1. Given a solution $f \in S_P(X \setminus \sigma)$, we consider the function f' which equals f in $U \setminus \sigma$ and is 0 in $X \setminus \overline{U}$. Then f' is a solution of the system $Pf' = 0$ with singularities on K'. Hence by Lemma 1.2 there exist a solution $f'_e \in S_P(X)$ and a sequence $\{c'_\alpha\}_{\alpha \in \mathbb{N}_0^n} \subset [L^p(K', m')]^k$, satisfying $\|\alpha! c'_\alpha\|^{1/|\alpha|}_{L^p(K',m')} \to 0$ when $|\alpha| \to \infty$, such that

$$f'(x) = f'_e(x) + \sum_{\alpha \in \mathbb{N}_0^n} \int_{K'} D^\alpha_x \Phi(x,y) c'_\alpha(y) dm'(y) \;\; (x \in X \setminus K').$$

Set $c_\alpha := c'_\alpha \mid_K,\ \alpha \in \mathbb{N}_0^n$. Since $\|\alpha! c_\alpha\|^{1/|\alpha|}_{L^p(K,m)} \to 0$ when $|\alpha| \to \infty$, the function f_e defined by

$$f_e(x) = f(x) - \sum_{\alpha \in \mathbb{N}_0^n} \int_K D^\alpha_x \Phi(x,y) c_\alpha(y) dm(y) \;\; (x \in X \setminus \sigma)$$

belongs to $S_P(X \setminus \sigma)$ because of Lemma 1.1. Moreover, this function satisfies the equation $Pf_e = 0$ also in a neighborhood of each interior point of K, since we have

$$f_e(x) = f'_e(x) + \sum_{\alpha \in \mathbb{N}_0^n} \int_{K' \setminus K} D^\alpha_x \Phi(x,y) c'_\alpha(y) dm'(y) \text{ for } x \in U \setminus \sigma.$$

Thus $f_e \in S_P((X \setminus \sigma) \cup \overset{\circ}{K})$, as was to be proved. □

The proof of Lemma 1.2 needs some preparation which we give in the following section by studying more thoroughly the topology on $S_{P'}(K)$. For the sake of simplicity, we restrict the following considerations to the case $k = 1$.

2. Equivalent topologies in $S_{P'}(K)$

2.1. Let K be any compact set in X. In this section, we study various topologies on $S_{P'}(K)$, where P' is the transposed operator to P. Define the space $S_{P'}(K)$ as follows. The function g belongs to $S_{P'}(K)$ if there exists an open set $U \supset K$ such that g is a solution of the equation $P'g = 0$ in U. If two such functions agree on some neighborhood of K, we identify them as elements in $S_{P'}(K)$.

For each U as above, let $S_{P'}(U)$ denote the space of solutions of the equation $P'g = 0$ in U with the topology of uniform convergence on compact subsets, i.e., the topology induced from $C(U)$. There is a natural map from $S_{P'}(U)$ into $S_{P'}(K)$, and we endow $S_{P'}(K)$ with the finest locally convex topology for which all these maps are continuous. We denote this topology by τ. Alternatively, the space $(S_{P'}(K), \tau)$ can be described as the inductive limit of the spaces $S_{P'}(U_\nu)$, where $\{U_\nu\}$ is any decreasing sequence of open sets containing K such that each neighborhood of K contains some U_ν, and such that each component of each U_ν meets $U_{\nu+1}$.

Remark 2.1. The space $(S_{P'}(K), \tau)$ is separated, a subset of this space is bounded iff it is contained and bounded in some $S_{P'}(U_\nu)$, and each closed bounded subset is compact. Proofs could be given by the same methods as in Koethe [9], p.379.

2.2. We will embed $S_{P'}(K)$ algebraically in a space $L^{(q)}$ whose topological dual consists of sequences of functions from $L^p(K, m)$. Lemma 1.2 follows from the Hahn-Banach Theorem once we show that the topology of $L^{(q)}$ restricted to $S_{P'}(K)$ is finer than the topology τ. To do this, we have first to study some Banach spaces.

Definition 2.1. Given positive numbers q and r, the space $l^q(r)$ is defined to consist of all sequences $\{\eta_\alpha\}_{\alpha\in\mathbb{N}_0^n} \subseteq \mathbb{C}$ with $(\sum_{\alpha\in\mathbb{N}_0^k} |\eta_\alpha|^q r^{|\alpha|q})^{1/q} < \infty$.

If K is an arbitrary compact subset of X and m is an arbitrary measure on K, then we denote by $l^q(r)^K$ the space of all functions $\eta(\cdot) = \{\eta_\alpha(\cdot)\}_{\alpha\in\mathbb{N}_0^n}$ on K with values in $l^q(r)$ such that $\eta_\alpha(\cdot) \in L^q(K, m)$ for every $\alpha \in \mathbb{N}_0^n$ and

$$(\sum_{\alpha\in\mathbb{N}_0^n} \|\eta_\alpha\|^q_{L^q(K,m)} r^{q|\alpha|})^{1/q} < \infty.$$

Lemma 2.1. *For $q \in [1, \infty]$, the functional*

$$\|\{\eta_\alpha\}\|_{l^q(r)^K} = (\sum_{\alpha\in\mathbb{N}_0^n} \|\eta_\alpha\|^q_{L^q(K,m)} r^{q|\alpha|})^{1/q} \tag{1}$$

defines a norm on $l^p(r)^K$.

Proof. The proof is an easy exercise from functional analysis. □

Equipped with the norm (1), the space $l^q(r)^K$ is a Banach space, provided $q \in [1, \infty]$. Instead of proving this directly, we proceed by the following

Lemma 2.2. *Let $r > 0, q \geq 1$ be arbitrary real numbers, and let $p \in]1, \infty]$ be the conjugate exponent to q. We have an isometrical isomorphism*

$$(l^q(r)^K)' \cong l^p(\frac{1}{r})^K.$$

Proof. Assume that $q > 1$. Fix some $\theta = \{\theta_\alpha\}_{\alpha\in\mathbb{N}_0^n} \in l^p(\frac{1}{r})^K$. Then θ defines a linear functional on $l^q(r)^K$ via $\langle\theta, \eta\rangle = \sum_{\alpha\in\mathbb{N}_0^n} \int_K \langle\theta_\alpha(y), \eta_\alpha(y)\rangle dm(y)$, for $\eta = \{\eta_\alpha\} \in l^q(r)^K$. Since

$$\begin{aligned} |\langle\theta, \eta\rangle| &\leq \sum_{\alpha\in\mathbb{N}_0^n} (\|\theta_\alpha\|_{L^p(K,m)} r^{-|\alpha|})(\|\eta_\alpha\|_{L^q(K,m)} r^{|\alpha|}) \\ &\leq \|\theta\|_{l^p(\frac{1}{r})^K} \cdot \|\eta\|_{l^q(r)^K}, \end{aligned} \tag{2}$$

this functional is continuous. Conversely, let $F \in (l^q(r)^K)'$. Given a multi-index $\alpha \in \mathbb{N}_0^n$, denote by e_α the element in $l^q(r)$ which is 1 in the α-th entry and 0 in all other entries. On $L^q(K, m)$, we may define a functional by juxtaposition $g \longmapsto F(ge_\alpha)$ for $g \in L^q(K, m)$. Since F is continuous, this functional is continuous, too. By duality, there is a function $\theta_\alpha \in L^p(K, m)$ such that $F(ge_\alpha) = \int_K \langle\theta_\alpha(y), g(y)\rangle dm(y)$ for all $g \in L^q(K, m)$. Since for an element $\eta = \{\eta_\alpha\}$ in $l^q(r)^K$ we have $\eta = \sum_{\alpha\in\mathbb{N}_0^n} \eta_\alpha e_\alpha$ and the series converges in the norm of $l^q(r)^K$, it follows that

$$F(\eta) = \sum_{\alpha\in\mathbb{N}_0^n} F(\eta_\alpha e_\alpha) = \sum_{\alpha\in\mathbb{N}_0^n} \int_K \langle\theta_\alpha(y), \eta_\alpha(y)\rangle dm(y).$$

Put $\theta := \{\theta_\alpha\}_{\alpha\in\mathbb{N}_0^n}$. To complete the proof, it remains to show that θ is in $l^p(\frac{1}{r})^K$. To this end, we consider the sequence $\{\eta_\alpha\}_{\alpha\in\mathbb{N}_0^n}$ of measurable functions on K given by

$$\eta_\alpha := \begin{cases} |\theta_\alpha|^{p-2}\overline{\theta}_\alpha r^{-p|\alpha|}, & \theta_\alpha \neq 0, \\ 0, & \theta_\alpha = 0. \end{cases}$$

Since $|\eta_\alpha|^q = |\theta_\alpha|^p r^{-pq|\alpha|}$ each function $\eta_\alpha(\cdot)$ is in $L^q(K,m)$. Hence it follows

$$\begin{aligned} \sum_{|\alpha|\le N} \|\theta_\alpha\|^p_{L^p(K,m)} (\frac{1}{r})^{p|\alpha|} &= |F(\sum_{|\alpha|\le N} \eta_\alpha e_\alpha)| \le \|F\|_{(l^q(r)^K)'} \| \sum_{|\alpha|\le N} \eta_\alpha e_\alpha\|_{l^q(r)^K} \\ &= \|F\|_{(l^q(r)^K)'} (\sum_{|\alpha|\le N} r^{-p|\alpha|} \|\theta_\alpha\|^p_{L^p(K,m)})^{1/q}. \end{aligned}$$

Thus $(\sum_{|\alpha|\le N} r^{-p|\alpha|}\|\theta_\alpha\|^p_{L^p(K,m)})^{1/p} \le \|F\|_{(l^q(r)^K)'}$ for every positive integer N. Together with (2) it follows that

$$\|\theta\|_{l^p(\frac{1}{r})^K} = \|F\|_{(l^q(r)^K)'},$$

as was to be proved.

For $q = 1$, the proof follows the same lines with the obvious modifications. □

Since the dual space to a normed space is a Banach space, Lemma 2.2 implies the following

Corollary 2.1. *Let $r > 0$ and $q > 1$. Then $l^q(r)^K$ is a reflexive Banach space.*

2.3. Note that if $r' > r'' > 0$, we have a continuous embedding $l^q(r')^K \hookrightarrow l^q(r'')^K$. Now let $\{r_\nu\}_{\nu\in\mathbb{N}}$ be some decreasing sequence of positive numbers tending to zero. The space $L^{(q)}$ is defined to be the inductive limit of the spaces $l^q(r_\nu)^K$. The space $L^{(q)}$ is separated. Each bounded set is contained and bounded in one of the $l^q(r_\nu)^K$. Moreover, $L^{(q)}$ is a (DF)-space, because it is the separated inductive limit of a sequence of normed, hence (DF)-, spaces (see Théorème 9 of Grothendieck [6]).

Our aim is to show that $S_{P'}(K)$ is topologically isomorphic to a subspace of $L^{(q)}$. Thus we proceed by constructing an embedding $S_{P'}(K) \hookrightarrow L^{(q)}$. More precisely, for each solution $g \in S_{P'}(K)$ we define

$$j(g) := \{\frac{D^\alpha g}{\alpha!}\mid_K\}_{\alpha\in\mathbb{N}_0^n}. \tag{3}$$

Lemma 2.3. *For every $g \in S_{P'}(K)$, the sequence $j(g)$ is in $L^{(q)}$, and the mapping $j : S_{P'}(K) \to L^{(q)}$ is continuous and injective.*

Proof. Let $g \in S_{P'}(K)$. Then there is a neighborhood U of K in X such that $g \in S_{P'}(U)$. Now choose a function $\varphi \in \mathcal{D}(X)$ which is equal to 1 in a neighborhood of K. Since Φ is a left fundamental solution of P, we get $g = \Phi' P'(\varphi g)$ in a neighborhood of K.

The function $P'(\varphi g)$ is supported by the closure of the set of those points $x \in U$ such that $\operatorname{grad}\varphi(x) \neq 0$. Let us denote this closure by σ. Then σ is a compact subset of $U \setminus K$, so there is a function $\psi \in \mathcal{D}(U\setminus K)$ which equals 1 in a neighborhood of σ.

Since $P'(\varphi g) = \psi P'(\varphi g)$, we have $g = \Phi'(\psi P'(\varphi g))$ in a neighborhood of K. Hence it follows for each multi-index α that

$$D^\alpha g(y) = \int P(x,D)(\psi(x) D_y^\alpha \Phi(x,y)) \cdot (\varphi(x)g(x))dx \quad (y \in K).$$

Using estimate (2) with $\tilde{K} = \text{supp } \psi$, we get

$$\begin{aligned}\sup_{y\in K}|D^\alpha g(y)| &\le c' a^{|\alpha|+\text{order } P}(|\alpha| + \text{order } P)! \sup_{x\in\text{supp }\varphi}|g(x)| \\ &\le c''(a')^{|\alpha|}|\alpha|! \sup_{x\in\text{supp }\varphi}|g(x)|,\end{aligned}$$

where a' is any number larger than a, and the constant c'' does not depend on $g \in S_{P'}(U)$ and α. It now follows that

(4)

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}_0^n}\|\frac{D^\alpha g}{\alpha!}\|^q_{L^q(K,m)} r_\nu^{q|\alpha|} &\le (c'')^q m(K)\Big(\sum_{\alpha\in\mathbb{N}_0^n}((a')^{|\alpha|} r_\nu^{|\alpha|}\frac{|\alpha|!}{\alpha!})^q\Big) \sup_{x\in\text{supp }\varphi}|g(x)| \\ &= (c'')^q m(K)\Big(\sum_{j=0}^{\infty}(na'r_\nu)^{qj}\Big) \sup_{x\in\text{supp }\varphi}|g(x)|.\end{aligned}$$

Choose ν_0 large enough, such that $nar_{\nu_0} < 1$. Then (4) shows that $j(g) \in l^q(r_{\nu_0})^K$ as well as the continuity of the mapping $j : S_{P'}(U) \longrightarrow l^q(r_{\nu_0})^K$.

Since a linear operator from $S_{P'}(K)$ into a locally convex space is continuous if and only if its restriction to each $S_{P'}(U)$ is continuous (for a proof cf. Bourbaki [3]), it follows that the mapping $j : S_{P'}(K) \longrightarrow L^{(q)}$ is continuous.

To show that j is injective let $g \in S_{P'}(K)$ be such that $j(g) = 0$. This means that $D^\alpha g\,|_K \equiv 0$ in K for all $\alpha \in \mathbb{N}_0^n$, and hence, since g is real analytic, it follows $g \equiv 0$ in a neighborhood of K. □

2.4. Now put

$$S_{P'}^{(q)} := j(S_{P'}(K)) \subseteq L^{(q)}.$$

We endow this space with the topology induced by $L^{(q)}$. We want to show

Lemma 2.4. *Let K be a locally connected compact subset of X, and $q > 1$. Then $S_{P'}^{(q)}$ is a closed subspace of $L^{(q)}$.*

For the proof of Lemma 2.4 we shall use the following result:

Lemma 2.5. *Assume that $\{L_\nu\}$ is a sequence of reflexive Banach spaces, such that L_ν is continuously embedded in $L_{\nu+1}$ for all ν, and L is the inductive limit of the sequence. Then a vector subspace Σ of L is closed if and only if for all ν the intersection $\Sigma \cap L_\nu$ is closed in L_ν.*

Proof. See Makarov [11]. □

Proof (of Lemma 2.4). Using Lemma 2.5 it is sufficient to show that for each ν the subspace $S_{P'}^{(q)} \cap (l^q(r_\nu)^K)$ is closed in $l^q(r_\nu)^K$.

Assume that for a solution $g \in S_{P'}(K)$ the image $j(g)$ is in $l^q(r_\nu)^K$. Then for all points $y \in K$, except perhaps for a set of zero measure m, we have

$$\Big(\sum_{\alpha\in\mathbb{N}_0^n}|\frac{D^\alpha g(y)}{\alpha!}|^q r_\nu^{q|\alpha|}\Big)^{1/q} < \infty.$$

Since the measure m is supposed to be massive, this inequality holds for a set σ_g of points $y \in K$ which is dense in K. So

$$\limsup_{|\alpha|\to\infty}|\frac{D^\alpha g(y)}{\alpha!}|^{1/|\alpha|} \le \frac{1}{r_\nu} \quad \text{for all } y \in \sigma_g.$$

We shall construct a complex neighborhood U_ν of K into which all the elements of $j^{-1}(l^q(r_\nu)^K)$ have (single valued) holomorphic extensions. This is the only place where we use the local connectedness of K.

For each $y \in K$ choose a neighborhood O_y in $\mathbb{C}^n$ such that $O_y \subset \Delta(y, r_\nu)$ and such that $K \cap O_y$ is connected. This is possible, since K is assumed to be locally connected. Here $\Delta(y, r) = \{z \in \mathbb{C}^n : |z_i - y_i| < r \ (i = 1, \dots, n)\}$ is the polydisk in $\mathbb{C}^n$ with center y and radius r. Choose r_y such that $\Delta(y, 2r_y) \subset O_y$. Define $U_\nu = \bigcup_{y \in K} \Delta(y, r_y)$. Then U_ν is a neighborhood of K in $\mathbb{C}^n$.

Let $g \in j^{-1}(l^q(r_\nu)^K)$ and $z \in U_\nu$. Define $\tilde{g}(z) = \sum_\alpha \frac{D^\alpha g(y)}{\alpha!}(z - y)^\alpha$ where y is any point of σ_g such that $z \in \Delta(y, r_y)$. The series converges, since $|z_i - y_i| < \frac{1}{2r_\nu}$ for all $i = 1, \dots, n$. We have to show that $\tilde{g}(z)$ does not depend on y.

Suppose that $z \in \Delta(y', r_{y'}) \cap \Delta(y'', r_{y''})$, where $y', y'' \in \sigma_g$. Let $r_{y''} \le r_{y'}$. Then $|y''_i - y'_i| < r_{y'} + r_{y'} \le 2r_{y'}$ for all $i = 1, \dots, n$; hence $y'' \in \Delta(y', 2r_{y'}) \subset O_{y'}$. We conclude that both y' and y'' belong to the connected set $K \cap O_{y'}$. Let U be an open set in $\mathbb{C}^n$ containing K, into which g has a (single valued) holomorphic extension. Then $K \cap O_{y'} \subset U \cap \Delta(y', r_\nu)$, and we denote by O the component of the set on the right which contains y'. Obviously, y'' is in O, too. The equation $g(z) = \sum_\alpha \frac{D^\alpha g(y')}{\alpha!}(z - y')^\alpha$ is valid for all $z \in O$. Hence the series

$$\tilde{g}(z) = \sum_\alpha \frac{D^\alpha g(y'')}{\alpha!}(z - y'')^\alpha \quad \text{about } y''$$

is a rearrangement of the series

$$g(z) = \sum_\alpha \frac{D^\alpha g(y')}{\alpha!}(z - y)^\alpha \quad \text{about } y',$$

and uniqueness of $\tilde{g}(z)$ follows.

It is obvious that $\tilde{g}$ is holomorphic in U_ν. Moreover, it is easily verified that $\tilde{g}$ and g agree on $U_\nu \cap U$. We may assume that the coefficients of the differential operator P have holomorphic extensions to U_ν. Then $P'\tilde{g} \equiv 0$ in U_ν, since the function $P'\tilde{g}$ is holomorphic in U_ν and vanishes on an open subset of each component of U_ν.

Thus every solution $g \in j^{-1}(l^q(r_\nu)^K)$ has a (single valued) extension to the complex neighborhood U_ν of K. Now, let $\{\eta^{(j)}\}$ be a sequence in $S^{(q)}_{P'} \cap l^q(r_\nu)^K$ which converges to an element $\eta = \{\eta_\alpha\}$ in $l^q(r_\nu)^K$. We would like to prove that η is in $S^{(q)}_{P'} \cap l^q(r_\nu)^K$, too. By definition of $S^{(q)}_{P'}$, for every $j = 1, 2, \dots$ there is a $g_j \in S_{P'}(K)$ such that $\eta^{(j)}_\alpha = \frac{D^\alpha g_j}{\alpha!}\,|_K \quad (\alpha \in \mathbb{N}^n_0)$. Moreover, as was already proved, each element g_j is represented by a holomorphic function $g_j(z)$ in the complex neighborhood U_ν of K satisfying $P'g_j = 0$ there.

The convergence $\eta^{(j)} \to \eta$ in $l^p(r_\nu)^K$ means that

$$\lim_{j\to\infty} \Big(\int_K \sum_{\alpha \in \mathbb{N}^n_0} r_\nu^{q|\alpha|} \Big|\frac{D^\alpha g_j(y)}{\alpha!} - \eta_\alpha(y)\Big|^q dm(y)\Big)^{1/q} = 0.$$

Hence it follows that there exists a subsequence $\{g_{j_s}\}$ such that for all points $y \in K$, except for a set of zero measure m, we have

$$\lim_{j_s\to\infty} \Big(\sum_{\alpha \in \mathbb{N}^n_0} r_\nu^{q|\alpha|} \Big|\frac{D^\alpha g_{j_s}(y)}{\alpha!} - \eta_\alpha(y)\Big|^q\Big)^{1/q} = 0. \tag{5}$$

Since the measure m is massive, equality holds for a set σ of points $y \in K$ which is dense in K. We now use compactness of K to conclude the following. There are a finite number of points $y^{(1)}, \dots, y^{(n)}$ in σ and a positive $r < r_\nu$ such that K is contained in the union $U = \Delta(y^{(1)}, r) \cup \ldots \cup \Delta(y^{(n)}, r)$, and $\overline{U} \subset U_\nu$. Our purpose is to show that the sequence $\{g_{j_s}\}$ converges to some function g in $S_{P'}(U)$. Since the space $S_{P'}(U)$ is complete, it suffices to prove that this sequence is a Cauchy sequence in $S_{P'}(U)$, i.e., in each of the spaces $C(k)$, where k is a compact subset of U. Obviously, we may restrict ourselves to compact sets k lying in one of the polydisks $\Delta(y^{(1)}, r), \dots, \Delta(y^{(n)}, r)$.

Let k be a compact subset of $\Delta(y, r)$ where $\Delta(y, r)$ is one of the polydisks previously mentioned. Denote by d the distance from k to the n-skeleton of $\Delta(y, r)$, i.e., $\partial_n \Delta(y, r) = \{\zeta \in \mathbb{C}^n : |\zeta_i - y_i| = r \ (i = 1, \dots, n)\}$. The distance is taken in the polydisk-norm.

We may regard some branch of $(g_{j_s}(z) - g_{j_t}(z))^q$ in $\Delta(y, r)$ to yield a holomorphic function there. By Cauchy's Theorem we have for all $z \in \Delta(y, r)$:

(6)
$$(g_{j_s}(z) - g_{j_t}(z))^q = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial_n \Delta(y,r)} \frac{(g_{j_s}(\zeta) - g_{j_t}(\zeta))^q}{(\zeta_1 - z_1) \cdot \ldots \cdot (\zeta_n - z_n)} d\zeta_1 \wedge \ldots \wedge d\zeta_n.$$

The Taylor-series expansion for $(g_{j_s}(\zeta) - g_{j_t}(\zeta))$, centered at y, converges uniformly in the closure of $\Delta(y, r)$. So (6) implies for $z \in k$:

$$\begin{aligned} |g_{j_s}(z) - g_{j_t}(z)| &\leq (\frac{1}{(2\pi d)^n \int_{\partial_n \Delta(y,r)} |g_{j_s}(\zeta) - g_{j_t}(\zeta)|^q |d\zeta_1| \wedge \ldots \wedge |d\zeta_n|)^{1/q}} \\ &= (\frac{1}{(2\pi d)^n} \int_{\partial_n \Delta(y,r)} |\sum_{\alpha \in \mathbb{N}_0^n} \frac{D^\alpha (g_{j_s}(y) - g_{j_t}(y))}{\alpha!} (\zeta - y)^\alpha|^q |d\zeta_1| \wedge \ldots \wedge |d\zeta_n|)^{1/q}. \end{aligned}$$

Using Hölder's inequality and taking into account that $r < r_\nu$, we get

$$\begin{aligned} &\sup_{z \in k} |g_{j_s}(z) - g_{j_t}(z)| \\ &\quad \leq (\frac{1}{(2\pi d)^n} \int_{\partial_n \Delta(y,r)} (\sum_{\alpha \in \mathbb{N}_0^n} \frac{|(\zeta - y)^\alpha|^p}{|r_\nu^{|\alpha|}|^p})^{(q/p)} |d\zeta_1| \wedge \ldots \wedge |d\zeta_n|)^{1/q} \\ &\quad \cdot (\sum_{\alpha \in \mathbb{N}_0^n} |\frac{D^\alpha (g_{j_s}(y) - g_{j_t}(y))}{\alpha!} r_\nu^{|\alpha|}|^q)^{1/q} \\ &\quad = (\frac{r}{d})^{n/q} (\sum_\alpha (\frac{r}{r_\nu})^{p|\alpha|})^{1/p} (\sum_\alpha |\frac{D^\alpha g_{j_s}(y) - D^\alpha g_{j_t}(y)}{\alpha!}|^q r_\nu^{q|\alpha|})^{1/q} \\ &\quad \leq (\frac{r}{d})^{n/q} (\frac{r_\nu^p}{r_\nu^p - r^p})^{n/p} \{ (\sum_\alpha r_\nu^{|\alpha| q} |\frac{D^\alpha g_{j_s}(y)}{\alpha!} - \eta_\alpha(y)|^q)^{\frac{1}{q}} \ + \\ &\qquad + \ (\sum_\alpha r_\nu^{|\alpha| q} |\frac{D^\alpha g_{j_t}(y)}{\alpha!} - \eta_\alpha(y)|^q)^{\frac{1}{q}} \}. \end{aligned}$$

By (5) it follows that $\sup_{z \in k} |g_{j_s}(z) - g_{j_t}(z)| \to 0$ when both j_s and j_t tend to infinity. This is just what we wanted to prove. Thus, there is a solution $g \in S_{P'}(U)$ such that $g_{j_s} \to g$ in $S_{P'}(U)$. Because of Lemma 2.3, we obtain $\eta = j(g)$. Hence $\eta \in S_{P'}^{(q)}$, as was to be proved. $\square$

The main result of this section consists of the following.

Theorem 2.1. *Assume that K is a locally connected compact subset of X, and $q > 1$. Then the mapping $j^{-1} : S_{P'}^{(q)} \to S_{P'}(K)$ is continuous.*

Proof. The assertion follows from Lemma 2.4 and a version of the Open Mapping Theorem, but we prefer the direct proof. As was already mentioned, the mapping $j^{-1} : S_{P'}^{(q)} \to S_{P'}(K)$ is continuous, iff each restriction $j^{-1} : S_{P'}^{(q)} \cap l^q(r_\nu)^K$ is continuous (see Bourbaki [3]). Let $\{g_j\}$ be a sequence of $S_{P'}(K)$ such that the sequence $\{\frac{D^\alpha g_j}{\alpha!}\}_{\alpha \in \mathbb{N}_0^n}$ converges to zero in $l^q(r_\nu)^K$. By the same way as we proceeded in the proof of Lemma 2.4, we find a complex neighborhood U_ν of K such that every element g_j is represented by a holomorphic function $g_j(z)$ in U_ν satisfying $P' g_j = 0$ there.

Choose a positive $r < r_\nu$ such that the set $U = \bigcup_{y \in K} \Delta(y, r)$ is contained in U_ν together with its closure. Then we claim that $\{g_j\}$ tends to zero uniformly on compact subsets of U. In fact, otherwise there would exist a compact set $k \subset U$, an $\varepsilon > 0$ and a subsequence $\{g_{j_s}\}$ such that $\sup_{z \in k} |g_{j_s}(z)| \geq \varepsilon$ for all j_s. But then it follows just in the same way as in the proof of Lemma 2.4 that some subsequence of $\{g_{j_s}\}$ should tend to zero uniformly on compact subsets of U. This contradiction implies our statement. Hence $g_j \to 0$ in $S_{P'}(K)$, as was to be proved. $\square$

Combining Theorem 2.1 and Lemma 2.3, we obtain the

Corollary 2.2. *Under the conditions of Theorem 2.1, the mapping $j : S_{P'}(K) \to S_{P'}^{(q)}$ is a topological isomorphism of the space $(S_{P'}(K), \tau)$ onto the space $S_{P'}^{(q)}$ equipped with the topology induced by $L^{(q)}$.*

3. Proof of the main lemma and remarks

3.1. In order to prove Lemma 1.2, we shall use the fact that each solution $f \in S_P(X \setminus K)$ may be written as the sum of a solution in $S_P(X)$ and a solution in $S_P(X \setminus K)$ which is *regular at infinity.* The latter notion can be introduced as follows:

Denote by $\hat{X}$ the one point compactification of X, i.e., the union of X and the symbolic point ∞. The topology in $\hat{X}$ is defined by the following system of neighborhoods: If $x \in X$, then we take the usual neighborhood basis, and if $x = \infty$, then we take the family of complements of all compact subsets in X. Let U be a neighbourhood of ∞. A function $f \in S_P(U)$ which has the representation (in a neighborhood of ∞, possibly smaller than U) $f = \Phi(F)$, for some distribution F with compact support, in K, is called *regular at infinity.* Here $\Phi(F)$ is the value of the pseudo-differential operator Φ on F. For smooth functions F with compact support $\Phi(F)$ is defined by $\Phi(F) = \int_{\mathbb{R}^n} \Phi(\cdot, y) F(y) dy$. For distributions F with compact support, $\Phi(F)$ is defined by duality.

Of course, this notion depends on our particular choice of the fundamental solution Φ, while the space of solutions regular at infinity does not depend on Φ on the whole.

Let us denote by $S_P^{(r)}(X \setminus K)$ the subspace of $S_P(X \setminus K)$ consisting of the solutions regular at infinity.

Lemma 3.1. *For each compact set $K \subset X$, it follows that*

$$S_P(X \setminus K) = S_P(X) \oplus S_P^{(r)}(X \setminus K).$$

The sum on the right is topological.

Proof. Let G_P be a Green operator for P, i.e., a bidifferential operator of order $\mathrm{ord}(P) - 1$ on X with the property that $dG_P(g, f) = (\langle g, Pf\rangle_x - \langle P'g, f\rangle_x)\, dx$ for all g and f, which are smooth enough in X. Here $dx = dx_1 \wedge \ldots \wedge dx_n$. Given a solution $f \in S_P(X \setminus K)$, we define the functions f_e and f_r in the following way. Let $x \in X$. Choose an open set $U \subset\subset X$ with piecewise smooth boundary such that $K \subset U$ and $x \in U$. Set $f_e(x) = -\int_{\partial U} G_P(\Phi(x, \cdot), f)$. It follows from the Green formula that $f_e(x)$ does not depend on the particular choice of U. Obviously, $f_e \in S_P(X)$. Now let $x \in X \setminus K$. Let $U \subset\subset X$ be an open set with piecewise smooth boundary such that $K \subset U$ and $x \notin \overline{U}$. Set $f_r(x) = \int_{\partial U} G_P(\Phi(x, \cdot), f)$. Again, f_r does not depend on the choice of U. It is clear that $f_r \in S_P^{(r)}(X \setminus K)$. By the Green formula we get $f = f_e + f_r$. The rest of the proof is obvious. □

Thus, every solution $f \in S_P(X \setminus K)$ may be written in the form $f = f_e + f_r$, with $f_e \in S_P(X)$ and $f_r \in S_P^{(r)}(X \setminus K)$, and this representation is unique.

3.2. Given a solution $f \in S_P(X \setminus K)$, we define a linear functional F_f on $S_{P'}(K)$ as follows. Let $g \in S_{P'}(K)$. This means that there is a neighborhood U of K such that $g \in S_{P'}(U)$. Choose a new neighborhood U_g of K such that $U_g \subset\subset U$ and the boundary of U_g is piecewise smooth. Put

$$\langle F_f, g\rangle = \int_{\partial U_g} G_P(g, f) \;\; (g \in S_{P'}(K)). \tag{1}$$

It follows from the Green formula, that the value $\langle F_f, g\rangle$ does not depend on the particular choice of U_g. Moreover, F_f is a continuous linear functional on $S_{P'}(K)$.

Lemma 3.2. *If $f \in S_P(X \setminus K)$, then*

$$\langle F_f, \Phi(x, \cdot)\rangle = f_r(x) \;\; \textit{for } x \in X \setminus K. \tag{2}$$

Proof. In fact, if $x \in X \setminus K$, then $\Phi(x, \cdot)$ satisfies $P'\Phi(x, \cdot) = 0$ in the neighborhood $X \setminus \{x\}$ of K. So the left-hand side of (2) is well-defined. To finish the proof, it only remains to look at the proof of Lemma 3.1. □

3.3. We proceed now by applying Theorem 2.1. Therefore, we are interested in a representation of functionals $F \in (L^{(q)})'$, where $1 < q < \infty$.

Lemma 3.3. *Let $1 < q < \infty$ and p be the conjugate exponent to q. To each continuous linear functional F on $(L^{(q)})$ there is a sequence $f = \{f_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ in $L^p(K, m)$ such that $\|f_\alpha\|_{L^p(K,m)}^{1/|\alpha|} \to 0$ when $|\alpha| \to \infty$, such that*

$$\langle F, \eta\rangle = \sum_{\alpha \in \mathbb{N}_0^n} \int_K \langle f_\alpha(y), \eta_\alpha(y)\rangle dm(y) \;\; \textit{for all } \eta = \{\eta_\alpha\} \in L^{(q)}.$$

Proof. Let $\eta \in l^q(r_\nu)^K$. Then $\eta = \sum_{\alpha \in \mathbb{N}_0^n} \eta_\alpha e_\alpha$, and the series converges with respect to the norm of $l^q(r_\nu)^K$. Since F is a continuous functional on $L^{(q)}$, its restriction to each of the $l^q(r_\nu)^K$ is continuous, too. Therefore, we have $\langle F, \eta\rangle = \sum_\alpha \langle F, \eta_\alpha e_\alpha\rangle$ for all $\eta = \{\eta_\alpha\} \in L^{(q)}$. For a fixed multi-index α, we consider the linear functional on $L^q(K, m)$ defined by $g \longmapsto \langle F, ge_\alpha\rangle \quad (g \in L^q(K, m))$. This functional is obviously continuous, so by duality there is a function $f_\alpha \in L^p(K, m)$ such that $\langle F, ge_\alpha\rangle = \int_K \langle f_\alpha(y), g(y)\rangle dm(y)$ for all $g \in L^q(k, m)$. Hence $\langle F, \eta\rangle = \sum_\alpha \int_K \langle f_\alpha(y), \eta_\alpha(y)\rangle dm(y)$ for all $\eta \in L^{(q)}$. The expression on the right hand side of this equality is a continuous linear functional on $L^{(q)}$, and thus on each of the spaces $l^q(r_\nu)^K$. Hence it follows by Lemma 2.2 that $\{f_\alpha\} \in l^p(\frac{1}{r_\nu})^K$ for every ν. Then $\sum_\alpha (\|f_\alpha\|_{L^p(K,m)}^{1/|\alpha|} \frac{1}{r_\nu})^{p|\alpha|} < \infty$, showing that $\limsup_{|\alpha|\to\infty} \|f_\alpha\|_{L^p(K,m)}^{1/|\alpha|} \leq r_\nu$ for all ν. Since $r_\nu \to \infty$, the assertion follows. □

3.4. We now turn to the

Proof (of Lemma 1.2). Assume that $f \in S_P(X \setminus K)$. We consider the continuous linear functional F_f on $S_{P'}(K)$ given by formula (1). The composition $F = F_f \circ j^{-1}$ defines a linear functional on the space $S_{P'}^{(q)}$, as follows from Lemma 2.3. Because of Theorem 2.1, the functional F is continuous. By the Hahn-Banach Theorem, F can be continuously extended to the whole space $L^{(q)}$. According to Lemma 3.3, there exists a sequence $\{f_\alpha\}_{\alpha\in\mathbb{N}_0^n}$ in $L^p(K,m)$, satisfying $\|f_\alpha\|_{L^p(K,m)}^{1/|\alpha|} \to 0$ when $|\alpha| \to \infty$, such that

$$\langle F, j(g)\rangle = \sum_\alpha \int_K \langle f_\alpha(y), \frac{D^\alpha g(y)}{\alpha!}\rangle dm(y) \text{ for all } g \in S_{P'}(K).$$

Now putting $g = \Phi(x,\cdot)$, where x is a fixed point of $X \setminus K$, and using Lemma 3.2 we derive the assertion of Lemma 1.2 with $c_\alpha = f_\alpha/\alpha!$ $(\alpha \in \mathbb{N}_0^n)$, since

$$\langle F, j(\Phi(x,\cdot))\rangle = \langle F_f, \Phi(x,\cdot)\rangle = f_r(x) = f(x) - f_e(x).$$

□

3.5. When K is a single point, the representation asserted by Lemma 1.2 is just the Laurent expansion of f.

Corollary 3.1. *Let y_0 be a fixed point of X. Then for every solution $f \in S_P(X \setminus \{y_0\})$ there exist a solution $f_e \in S_P(X)$ and a sequence $\{c_\alpha\}_{\alpha\in\mathbb{N}_0^n} \subset \mathbb{C}^k$, satisfying $|\alpha! c_\alpha|^{1/|\alpha|} \to 0$ when $|\alpha| \to \infty$, such that*

$$f(x) = f_e(x) + \sum_\alpha D_y^\alpha \Phi(x, y_0) c_\alpha \quad (x \in X \setminus \{y_0\}). \tag{3}$$

Proof. The assertion follows by using $m(y_0) = 1$ as a massive measure on $K = \{y_0\}$. □

The coefficients $\{c_\alpha\}$ will not be uniquely determined by f, since

$$P'(y_0, D_y)\Phi(x, y_0) = \delta(x - y_0) I_k$$

becomes zero off y_0.

The Laurent-series expansions for solutions of general elliptic equations were first studied by Lopatinskii [10] .

3.6. If $O \subset\subset X$ is an open set whose boundary is locally connected, then each solution f of $Pf = 0$ in O has a representation (1) for $x \in O$ with $K = \partial O$. The only thing we have to do is to construct a massive measure m on ∂O, and to extend f to a function satisfying the equation in the complement of ∂O. The assertion follows by Lemma 1.2.

3.7. Theorem 1.1 implies that arbitrary singularities of solutions of elliptic equations may be locally separated into atomic (i.e. one-point) singularities.

Corollary 3.2. *Assume that K is a locally connected compact subset of σ, and $\{y_\nu\}$ is a dense sequence of points of K. Then every solution $f \in S_P(X \setminus \sigma)$ can be written in the form $f = f_e + \sum_\nu f_\nu$, where $f_e \in S_P((X \setminus \sigma) \cup \overset{\circ}{K})$ and $f_\nu \in S_P(X \setminus \{y_\nu\})$, and the series converges in the topology of $S_P(X \setminus K)$.*

Proof. We use the massive measure m on K constructed in Example 1.1. By Theorem 1.1

$$f(x) = f_e(x) + \sum_\alpha \Big(\sum_\nu D_y^\alpha \Phi(x, y_\nu) c_\alpha(y_\nu) \mu_\nu\Big) \text{ for } x \in X \setminus \sigma,$$

where $f_e \in S_P((X \setminus \sigma) \cup \overset{\circ}{K})$ and $\lim_{|\alpha| \to \infty} (\sum_\nu |\alpha! c_\alpha(y_\nu)|^p \mu_\nu)^{1/(p|\alpha|)} = 0$. The last condition allows to rearrange the summations and to derive $f = f_e + \sum_\nu f_\nu$ with

$$f_\nu = \sum_\alpha D_y^\alpha \Phi(x, y_\nu) c_\alpha(y_\nu) \mu_\nu,$$

as was to be proved. ☐

3.8. For the Laplace operator we obtain the following result (which seems to be new).

Corollary 3.3. *Let $K \subset \sigma$ be a locally connected compact set, and $1 < p < \infty$. Then every harmonic function f in $X \setminus \sigma$ has the form*

$$f(x) = f_e(x) + \sum_{j=0}^{\infty} \int_K \frac{h_j(y, x-y)}{|x-y|^{n+2(j-1)}} dm(y) \quad (x \in X \setminus \sigma)$$

where f_e is a harmonic function in $(X \setminus \sigma) \cup \overset{\circ}{K}$, and $h_j(y, z)$ are homogeneous harmonic polynomials of degree j in z with coefficients in $L^p(K, m)$ with respect to y, such that $\lim_{j \to \infty} (\frac{1}{j!} \int_K |h_j(y, D_z) h_j(y, z)|^{p/2} dm(y))^{1/p_j} = 0$.

Proof. It suffices to transform formula (1) by means of the Hecke identity (cf. Stein [14]). ☐

3.9. We finish this section by mentioning one more aspect of Theorem 1.1. It is a natural question to ask whether a given solution $f \in S_P(X \setminus \{y_0\})$ admits a representation (3) with a finite number of summands. This is obviously the case iff f has a finite order of growth near y_0, i.e., $|f(x)| \le c|x - y_0|^{-\gamma}$ in some deleted neighborhood of y_0. In other words, y_0 has to be a pole of f. Therefore, the solutions $f \in S_P(X \setminus K)$ for which the expansions (1) have only a finite number of terms are analogues of solutions with poles in general. Such solutions can be characterized as follows.

Theorem 3.1. *Let K be an arbitrary compact set in X, m be a massive measure on K, and $1 < p < \infty$. A solution $f \in S_P(X \setminus K)$ has a representation (1) with a finite number of terms iff the functional F_f given by (1) is continuous on $S_{P'}(K)$ with respect to the topology defined by the family of seminorms $\|D^\alpha g\|_{L^q(K,m)}$ $(\alpha \in \mathbb{N}_0^n)$.*

Proof. See Tarkhanov [15]. ☐

References

1. Baernstein, A.: Representations of holomorphic functions by boundary integrals. *Trans. Amer. Math. Soc.* **160** (1971), 27-37. MR **44:**415
2. Baernstein, A.: A representation theorem for functions holomorphic off the real axis. *Trans. Amer. Math. Soc.* **165** (1972), 159-165. MR **45:**2190
3. Bourbaki, N.: *Topological vector spaces.* Springer-Verlag, Berlin, Heidelberg, New York, 1987. MR **88g:**46002
4. Fischer, B.; Tarkhanov, N.N.: A representation of solutions with singularities. *Contemp. Math.*, vol. 212, Amer. Math. Soc., Providence, RI, 1998. CMP 98:05
5. Gramsch, B.: Über das Cauchy-Weil Integral für Gebiete mit beliebigem Rand. *Arch. Math. (Basel)* **28** (1977), 409-421. MR **58:**17206
6. Grothendieck, A.: Sur les espaces (F) and (DF). *Summa Brasil. Math.* **3** (1954), 57-123. MR **17:**765b
7. Havin, V.P.: An analogue of the Laurent series, in: *Investigations in modern problems of the theory of functions of a complex variable.* Fizmatgiz, Moscow 1961, 121-131 (Russian).
8. Havin, V.P.: Golubev series and the analyticity on a continuum, in: *Linear and complex analysis problem book.* Springer Lecture Notes 1043. Springer-Verlag, Berlin, Heidelberg, New York 1984. MR **85k:**46007
9. Köthe, G.: *Topologische lineare Räume I.* Springer-Verlag, Berlin, Heidelberg, New York, 1960. MR **24:**A411
10. Lopatinskii, Ya. B.: Behaviour of solutions of a linear elliptic system in a neighborhood of an isolated singular point. *Dokl. Akad. Nauk SSSR* **79** (1951) 5, 727-730 (Russian).
11. Makarov, B.M.: Inductive limits of normed spaces. *Dokl. Akad. Nauk SSSR* **119** (1958) 6, 1092-1094 (Russian). MR **20:**5412
12. Rogers, J.T.; Zame, W.R.: Extension of analytic functions and the topology in spaces of analytic functions. *Indiana Univ. Math. J.* **31** (1982) 6, 809-818. MR **83k:**30050
13. Simonova, S.: A representation theorem for functions harmonic off a hyperplane. *Sibirsk. Mat. Zh.* **34** (1993) (Russian).
14. Stein, E.M.: *Singular integrals and differentiability properties of functions.* Princeton University Press, Princeton 1970.
15. Tarkhanov, N.N.: The structure of solutions of elliptic systems with a compact set of singularities. *Izv. VUZ. Mat.* **1989** no. 12, 47-56 (Russian). MR **91e:**35090
16. Tarkhanov, N.N.: *Laurent series for solutions of elliptic systems.* Nauka, Novosibirsk 1991 (Russian). MR **94e:**35013
17. Varfolomeev, A.L.: Analytic continuation from a continuum onto its neighborhood; in: Zap. Nauchni. Sem. Leningrad. Otdel. Mat. Inst. Stekl. (LOMI) **113** (1981), 27-40 (Russian). MR **83e:**30003

Max-Planck-Arbeitsgruppe, "Partielle Differentialgleichungen und Komplexe Analysis", Universität Potsdam, Am Neuen Palais 10, D - 14415 Potsdam, Germany

E-mail address: `christoph@mpg-ana.uni-potsdam.de`

E-mail address: `tarkhan@mpg-ana.uni-potsdam.de`